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Space-filling Orthogonal Array based Composite Designs

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Abstract

An efficient design that can analyze high dimensional simulation model by using fewer runs as compared with other second-order designs is needed in a complex system. In this paper, a new class of designs called space-filling orthogonal-array based composite designs was proposed using the centered l_2-discrepancy and maximin distance. The proposed designs were constructed and compared with other existing composite designs such as orthogonal-array based composite designs, centered composite designs and small composite designs based on relative D-efficiency and D_s-optimality criterion for full model, linear, quadratic and bilinear terms respectively. The results from the comparison show that the space-filling orthogonal-array based composite designs are better in terms of efficiency and run sizes for some cases especially as the number of factors increases.

Keywords: Central composite designs, Space-filling, Response surface designs, Orthogonal array, Optimality criteria.

1 Introduction

In a complex nature of a system, analysts need an efficient design that can explore high dimensional simulation models. These simulation models are critical

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to the early phases of system design and involve complicated outputs with a wide variety of linear and nonlinear response surface forms, Cioppa (2002). Response surface designs such as orthogonal arrays (OAs) and composite designs (central composite design and small composite design) are used to model the effect of the individual factors in the second order model. In other to compliment the shortcoming of OAs and composite designs, Xu et al. (2014) constructed a design called orthogonal array composite design (OACD) that comprises of a two-level factorial portion, three-level OA as the axial portion and center points. A number of researchers had discussed OACD and such include; Xu et al. (2014) constructed OACDs which are effective for factor screening and response surface modeling, Zhou and Xu (2016) assessed the properties of the OACDs and definitive screening composite designs (DSCDs) and most recently, Chen et al. (2017) studied the robustness of OACD to missing observation using minimax loss criterion. Unfortunately, the OACD have significant limitations in that they cannot analyze simulations that represent complicated systems.

In order to explore the complex relationships between input and output variables, and to find an approximate model that is much simpler than the true but complicated model, a good space-filling design (Zhou and Xu, 2014) can be used to fix the bias between the approximate model and the true model by distributing the design points evenly over the experimental boundaries. In the literature, several space-filling criteria have been proposed and there are two broad categories: uniformity-based and distance-based criteria. Some of such include; Zhou and Xu (2014), Xiao and Xu (2017), Sukdaiphueng et al. (2017), Wu et al. (2017), Cioppa (2002), Luc Pronzato (2012), Cioppa and Lucas (2007), Joseph et al. (2018) and Joseph and Hung (2008). Space-filling are applied in response surface designs such as OACD of Xu et al. (2014) in order to fit a variety of different high-order models without making a priori assumptions about the response surface.

In this work, a new space-filling orthogonal-array based composite designs (SOACDs) were proposed which are based on a two-level full or fractional factorial design and a three-level orthogonal array (Xu et al., 2014 & Chen et al., 2017) with space-filling properties such as uniformity (Fang and Lin, 2003 & Androulakis et al., 2016) and maximin distance (Johnson et al., 1990) which enable the parameters to be estimated without loss of efficiency than in other composite designs such as central composite designs (CCDs) and small composite designs (SCDs) etc. The proposed designs were compared with the other composite designs such as orthogonal array based composite designs, centered composite design and small composite designs under the D-efficiency and D_s-optimality for full model, linear terms, quadratic terms, and bilinear terms, respectively.

2 Methodology

In this section, we give a brief discussion of all the composite designs; space-filling designs; Metamodels and space-filling criteria used in this paper.

2.1 Composite Designs

Central composite designs (CCD) are the most popular class of second-order response surface designs, which was introduced by Box and Wilson (1951). These designs are sequential in nature, which forms the basis of the sequential assembly of the response surface designs. These designs involve n_f runs of the two-level k-factor full factorial or fractional factorial of resolution V combined with the n_α axial or star points as a second part, and finally n_c center runs. The factorial portion of the CCD contributes to the estimation of linear and two-factor interaction terms, the axial part contributes to the estimation of the quadratic terms in the model while the center runs provides some extra degrees-of-freedom to estimate the pure error. In order to reduce the run size, Draper and Lin (1990) proposed the small composite designs (SCDs) by using Plackett-Burman (1946) designs (PBDs) as the factorial portion. In both CCD and SCD, the axial portion consisting of $2k$ points arranged so that two points are chosen on the coordinate axis of each variable at a distance of $\alpha$from the design center. An orthogonal-array based composite design (OACD) proposed by (Xu et al., 2014) is a composite design that comprises of a two-level factorial portion, denoted by n_f and the three-level orthogonal-array axial portion denoted by n_α. The two-level part was used to estimate the linear effects and two-factor interactions between variables while the three-level orthogonal array was used to estimate the linear and quadratic effect. The OACD differs from the CCD or SCD in the way they choose the additional points. The CCD or SCD employs a one-factor-at-a-time approach for the additional points because each axial point has only one nonzero component. However, the axial points of a CCD or SCD provide no information on bilinear (or interaction) terms and resolution IV designs cannot be used as the two-level portion. For this reason, the SCD must use a resolution III design as the factorial part even if a resolution IV design with the same size exists. An OA of N runs, k columns, s levels and strength t, denoted by OA(N, s^k, t), is an $N \times k$ matrix in which for each pair of columns, its combination of levels occurs equally often, Hedayat et al. (1999). OACD can use resolution IV designs as the two-level portion which is important for factor screening and in a sequential experiment and also, its attractive feature is that it allows us to perform multiple analyses with different parts of the data for cross validation, Xu et al. (2014).

2.2 Space-filling Designs

Space-filling designs aim at spreading the design points throughout the experimental region as evenly as possible. Some of the popular space-filling designs include Latin hypercube designs (McKay et al., 1979), uniform designs (Fang and Wang, 1994), distance-based designs such as maximin and minimax (Johnson et al., 1990) and so on. These designs are robust in modeling decisions and are therefore commonly used as designs for computer experiments. Uniformity in design ensures that the points within the experimental region are distributed throughout the interior of the cube and is not limited to the corners or surfaces of the cube, Cioppa (2002). Discrepancies are used to measure the departure of uniformity of a given design and all have their geometrical meanings and can be expressed as the difference between the empirical distribution and the uniform distribution, Tang et al. (2012). Centered l_2-discrepancy (CD) Hickernell (1998) which is invariant under reordering the runs, relabeling factors, reflections of the set of design points about any plane passing through the center of the unit hypercube and parallel to its faces, Fang and Mukerjee, (2000). It was observed that designs with low discrepancy tend to have good space-filling properties and are model robust (Zhou and Xu, 2014) in the sense that they can guard against incorrect estimates due to model misrepresentation. Maximin (Mm) distance (Johnson et al., 1990), is an important space-filling criterion widely used in computer experiment to measure how uniformly the design points scatter over an experimental region so that the separation distance is maximized reasonably for prediction anywhere in the domain.

2.3 Metamodels

A good Metamodel is one in which response makes close use of the variables available and the errors are very small, Cioppa and Lucas (2007). The most commonly used metamodel is one in which y is a linear combination of the inputs, that is;

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \varepsilon_i \tag{1}$$

In computer simulations with many input variables that gives responses, a linear metamodel may not sufficiently characterize the response surface of a high-order polynomial that include a curvilinear and interaction terms. In the presence of high-order polynomials, multicollinearity among the input factors can affect the precision of the estimates. To alleviate the problem, lower-order orthogonal polynomial given by,

$$y(x) = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i=1}^{k-1} \sum_{j>i}^{k} \beta_{ij} x_i x_j + \varepsilon_i \quad (2)$$

have been introduced to model computer experiments where k is the number of independent factors,β_0,β_i,β_{ii}andβ_{ij} are the intercept, linear, quadratic and bilinear (or interaction) terms respectively, and $\varepsilon_i$is the random with mean 0, variance 1 and independence between any pair of runs. The number of simulations, n_s runs must satisfy$n_s \geq k + k + \binom{k}{2} + 1$to have sufficient degrees of freedom to estimate the coefficients in the model. However, in practice, only a small percentage of the input factors are generally significant.

2.4 Space-filling criteria

Hickernell (1998) defined squared centered l_2-discrepancy (CD^2) of a design D over the unit cubeC^n= [0,1] ^{n} as:

$$CD^2(D) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \prod_{k=1}^{n} \left(1 + \frac{1}{2}\left|u_{ik} - \frac{1}{2}\right| + \frac{1}{2}\left|u_{jk} - \frac{1}{2}\right| - \frac{1}{2}\left|u_{ik} - u_{jk}\right|\right) - \frac{2}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \left(1 + \frac{1}{2}\left|u_{ik} - \frac{1}{2}\right| - \frac{1}{2}\left|u_{ik} - \frac{1}{2}\right|^2\right) + \left(\frac{13}{12}\right)^n \quad (3)$$

where $u_{ik} = (2x_{ik} + 1)/(2s)$,$0<u_{ik}<1$ and $0 \leq x_{ik} \leq s-1$, s represents levels, n is the number of input parameters that is the dimension of the design space, u_{ik} and $u_{jk}$are thek^{th} coordinates of the i^{th} and j^{th} points respectively. However, the wordlength pattern remains the same for combinatorial isomorphic designs, Tang et al. (2012) since these designs are obtain from permuting the levels in the same column whereas its centered $l_2$discrepancy will not be the same when levels of factors are permuted.

Maximin distance criterion, Johnson et al. (1990), measured how uniformly the experimental points are scattered through the domain such that no two points are close to each other. Let$u = \{u_1, ..., u_n\}$and $v = \{v_1, ..., v_n\}$be two design points in the design space$[0,1]^k$ for k factors. For$t$>0, define the inter-site distance $u$and $v$to be;

$$d(u,v) = \{\sum_{i=1}^{k} |u_i - v_i|^t\}^{1/t} \quad (4)$$

when$t$ =1 and t =2, the measure in (4) becomes the rectangular and euclidean distances, respectively. The maximin distance criterion seeks a design d of n points in the design space$[0,1]^k$ that maximizes

$$\min_{u,v \in d} d(u,v) \quad (5)$$

where$u \neq v$and $d(u,v)$ is defined in (4) for any given t.

3 Construction of Space-filling Orthogonal-Array based Composite Design (SOACD)

In this section, the steps adopted in constructing the SOACD are presented. A general guideline is to use a two-level fractional factorial, and three-level orthogonal array in (Xu et al., 2014) shown in Table 1, generalized minimum aberration (GMA) in screening out poor designs and level permutations of the column of the GMA designs to improve the space-filling property of the design.
Firstly, a two-level fractional factorial and three-level orthogonal array were selected (strength two) and the GMA criterion was used to find the best sub-design with n columns. For small designs, all level permutations can be conducted to find the best design under CD and maximin criteria whereas for larger designs, it may be infeasible to perform all possible level permutation when the number of levels of the design and the number of runs increase. In such circumstance, a local search heuristic stochastic algorithm proposed by Tang and Xu (2013) can be used to find optimal or near optimal level permutation.

Algorithm1: **Pseudocode for prototype local search heuristic**
Initialize τ (number of iterations)
Initialize δ (impairment threshold) and u_o (acceptance probability)
Input starting design D_s and let $D_{\min} := D_s$
for $i = 1$ to τ **do**
  Generate $D_{new} \in D_s$ (neighbour to current solution)
  Compute $\nabla = CD^2(D_{new}) - CD^2(D_s)$ and generate a random number u from $u[0,1]$
  if($\nabla < 0$) or ($\nabla < \delta$ and $u < u_0$):
  then let $D_s := D_{new}$
  else if $CD^2(D_s) < CD^2(D_{\min})$:
  then let $D_{\min} := D_s$
end for

Algorithm 2:
2-level
Step 1. Choose a 2^k full factorial ($k \geq 5$) or a regular $2^{k\,--\,p}$ fractional factorial orthogonal design.
Step 2. Use the GMA criterion to select the best design from some exiting designs.
Step 3. From the GMA design selected, conduct all possible level permutation and select the design with the least discrepancy. h_2 is the two-level design selected with n_f runs.
3-level OA
Repeat *Step1* to *Step 3.* h_3 is the three-level design selected (1, 0, -1) with n_α runs.

Step 4. Combine $h_2$and$h_3$together to form a space-filling orthogonal-array based composite design (SOACD)$h' = \begin{pmatrix} h_2 \\ h_3 \end{pmatrix}$.

The starting design is the resulting GMA design from *Step 2*. This helps to reduce the complexity but restricts the structure of the starting design. Tang and Xu (2013) defined the neighbour to current solution, $N(D_s)$ by exchanging all elements of two distinct levels within the same column of the current design D_s so that the combinatorial structure of the design is unchanged. The impairment threshold δ, which is set to be one-tenth of the squared discrepancy of the starting design, is around 0.09 for the two-level and 0.02 for the three-level while the acceptance probability u_o is between 0.1 and 0.3. CD^2 was used as the space-filling measure in *Step 3* in *Algorithm 2* while maximin criterion also gives similar result. In each iteration, the algorithm will generate a new candidate solution D_{new} in the neighborhood of the current solution D_s. If this new candidate solution results in an improved value of the CD^2, that is; if $\nabla < 0$ or ($\nabla < \delta$and $u < u_0$)the new candidate solution becomes the current solution.

Preposition: The properties of the resulting design depend on which two-level column that is aligned with which three-level column. Although level permutations change geometrical structures, all designs evaluated in *Step 3* have the same generalized wordlength pattern (GWP) as its starting GMA designs since they are combinatorial isomorphic.

Table 1: Two-level factorial and Three-level OA for different values of = 5, 6, ...

	Two-level factorial			**Three-level OA**	
k	**Designs**	**n**	**Column and generators**	**Designs(n_a)**	**Column**
5	2^{5-1}_{V}	16	E = ABCD	OA(18)	(2-6)
5	PB(12)	12	(1-5)	OA(18)	(2,5,3,4,6)
5	2^{5-2}_{III}	8	D=ABC;E=AB	OA(18)	(2-4,5,6)
6	2^{6-1}_{VI}	32	F=ABCDE	OA(18)	(1-6)
6	PB(20)	20	(1-5,13)	OA(18)	(1,4,6,3,2,5)
6	PB(12)	12	(1-5,7)	OA(18)	(2,5,3,4,6,1)
7	2^{7-1}_{VII}	64	G=ABCDEF	OA(18)	(1-7)
7	2^{7-2}_{IV}	32	F=ABCD;G=ABE	OA(18)	(1,2,5,3,4,7,6)
7	PB(20)	20	(1-5,13,16)	OA(18)	(3,1,5,7,4,2,6)
8	2^{8-2}_{V}	64	G = ABCDE;H = ABCF	OA(27)	(1-8)
8	2^{8-3}_{IV}	32	F = ABCD;G = ABE;H = ACE	OA(27)	(1,3,4,5,2,7,8,6)
8	PB(20)	20	(1-5,13,16,15)	OA(27)	(6,3,8,4,2,1,7,5)
9	2^{9-2}_{V}	128	H = ABCDE; J = ABCFG	OA(27)	(1-9)
9	2^{9-3}_{IV}	64	G = ABCDE;H = ABCF; J = ADF	OA(27)	(1,3,8,2,6,7,5,4,9)
9	2^{9-4}_{IV}	32	F = ABCD;G = ABE;H = ACE; J = ADE	OA(27)	(5,6,1,7,2,4,9,3,8)
10	2^{10-3}_{V}	128	H = ABCDE; J = ABCFG;K = ABDF	OA(27)	(1-10)
10	2^{10-4}_{IV}	64	G = ABCDE;H = ABCF; J = ADF;K = ABEF	OA(27)	(5,6,8,2,3,4,10,7,9,1)

The 2-level factorial and 3-level OA for different values of *k* are from Xu et al. (2014)

Table 2: Structure of the optimization algorithm for two-level

D_s						D_{new}				
X1	X2	X3	X4	X5		X1	X2	X3	X4	X5
1	-1	-1	-1	-1		1	-1	-1	-1	**1**
-1	1	-1	-1	-1		-1	1	-1	-1	**1**
-1	-1	1	-1	-1		-1	-1	1	-1	**1**
-1	-1	-1	1	-1		-1	-1	-1	1	**1**
-1	-1	-1	-1	1		-1	-1	-1	-1	**-1**
1	1	1	-1	-1		1	1	1	-1	**1**
1	1	-1	1	-1		1	1	-1	1	**1**
1	1	-1	-1	1		1	1	-1	-1	**-1**
1	-1	1	1	-1	→	1	-1	1	1	**1**
1	-1	1	-1	1		1	-1	1	-1	**-1**
1	-1	-1	1	1		1	-1	-1	1	**-1**
-1	1	1	1	-1		-1	1	1	1	**1**
-1	1	-1	1	1		-1	1	-1	1	**-1**
-1	1	1	-1	1		-1	1	1	-1	**-1**
-1	-1	1	1	1		-1	-1	1	1	**-1**
1	1	1	1	1		1	1	1	1	**-1**

Table 3: Structure of the optimization algorithm three-level

D_s						D_{new}				
X1	X2	X3	X4	X5		X1	X2	X3	X4	X5
-1	-1	-1	-1	-1		-1	**1**	-1	-1	-1
0	0	0	0	0		0	**0**	0	0	0
1	1	1	1	1		1	**-1**	1	1	1
-1	-1	0	0	1		-1	**1**	0	0	1
0	0	1	1	-1		0	**0**	1	1	-1
1	1	-1	-1	0		1	**-1**	-1	-1	0
-1	0	-1	1	0		-1	**0**	-1	1	0
0	1	0	-1	1	→	0	**-1**	0	-1	1
1	-1	1	0	-1		1	**1**	1	0	-1
-1	1	1	0	0		-1	**-1**	1	0	0
0	-1	-1	1	1		0	**1**	-1	1	1
1	0	0	-1	-1		1	**0**	0	-1	-1
-1	0	1	-1	1		-1	**0**	1	-1	1
0	1	-1	0	-1		0	**-1**	-1	0	-1
1	-1	0	1	0		1	**1**	0	1	0
-1	1	0	1	-1		-1	**-1**	0	1	-1
0	-1	1	-1	0		0	**1**	1	-1	0
1	0	-1	0	1		1	**0**	-1	0	1

Table 4: Structure for 34-runs of SOACD with five factors and nc = 0

X1	X2	X3	X4	X5
1	-1	-1	-1	1
-1	1	-1	-1	1
-1	-1	1	-1	1
-1	-1	-1	1	1
-1	-1	-1	-1	-1
1	1	1	-1	1
1	1	-1	1	1
1	1	-1	-1	-1
1	-1	1	1	1
1	-1	1	-1	-1
1	-1	-1	1	-1
-1	1	1	1	1
-1	1	-1	1	-1
-1	1	1	-1	-1
-1	-1	1	1	-1
1	1	1	1	-1
-1	1	-1	-1	-1
0	0	0	0	0
1	-1	1	1	1
-1	1	0	0	1
0	0	1	1	-1
1	-1	-1	-1	0
-1	0	-1	1	0
0	-1	0	-1	1
1	1	1	0	-1
-1	-1	1	0	0
0	1	-1	1	1
1	0	0	-1	-1
-1	0	1	-1	1
0	-1	-1	0	-1
1	1	0	1	0
-1	-1	0	1	-1
0	1	1	-1	0
1	0	-1	0	1

Table 1 represents two-level factorial and three-level OA for different values of k, while Tables 2 and 3 represent the structure of the optimization algorithm for two-level factorial and three-level orthogonal GMA designs, respectively. The combination of the candidate solution D_{new} in Tables 2 and 3 gives the SOACD in Table 4 and it is optimal with respect to centered l_2-discrepancy and maximin criterion under the second-order model.

4 Comparison of Results

In this section, the proposed SOACDs were compared with other composite designs (CCDs, SCDs and OACDs) in terms of relative D-efficiency and D_s-optimality.

4.1 Comparison based on relative $D-$efficiency

The relative *D*-efficiency of SOACDs were compared with the corresponding CCDs, SCDs and OACDs for $5 \leq k \leq 11$ and $1 \leq n_c \leq 5$. The results are presented in Table 5. The relative *D*-efficiency (D_{r-eff}) is defined to be

$$D_{r-eff} = \left(\frac{|X^T X|_A}{|X^T X|_B} \right)^{1/p} \tag{6}$$

where $p = (k + 1)(k + 2)/2$; *A* is the proposed SOACD; and *B* is any of the composite designs.

Remark: *D*-efficiency greater than one implies that Design *A* is better than Design *B*.

Table 5: Two-level factorial and Three-level OA for different values of = 5, 6, . . .

		Number of factors (*k*)						
n_c	Design	5	6	7	8	9	10	11
1	**CCD**	0.8468	0.8432	0.9674	1.1278	1.1583	1.1827	1.2034
	SCD	0.9563	0.9857	1.0162	1.0817	1.1673	1.1878	1.2187
	OACD	1.0457	1.1054	1.2061	1.1873	1.2114	1.2835	1.3358
2	**CCD**	0.8541	0.8482	1.0621	1.1794	1.2158	1.2437	1.2753
	SCD	0.9793	1.0278	1.0673	1.0924	1.1583	1.2054	1.2267
	OACD	1.0832	1.1326	1.1321	1.2011	1.2439	1.3015	1.3512
3	**CCD**	0.9237	1.1614	1.2389	1.2529	1.2653	1.2784	1.2976
	SCD	1.0874	1.1247	1.1693	1.1983	1.2372	1.2521	1.2787
	OACD	1.1280	1.1458	1.1761	1.2459	1.2761	1.3422	1.3634
4	**CCD**	0.9672	0.9892	1.1625	1.2386	1.2704	1.2804	1.3011
	SCD	1.1673	1.1869	1.2045	1.2189	1.2369	1.2687	1.2930
	OACD	1.1324	1.1654	1.2474	1.2731	1.2841	1.3541	1.3745
5	**CCD**	0.9734	0.9967	1.2563	1.2689	1.2945	1.3123	1.3345
	SCD	1.1789	1.1838	1.2035	1.2478	1.2732	1.2943	1.3030
	OACD	1.1521	1.1723	1.2873	1.2843	1.2912	1.3752	1.3848

From Table 5, it was observed that *D*- efficiency of SOACDs is better than OACDs as the factors increases for $5 \leq k \leq 11$ and $1 \leq n_c \leq 5$. Furthermore, it can be seen that SOACDs perform relatively better than the CCDs and SCDs for all cases except for where relative *D*-efficiency is less than 1.

4.2 Comparison based on the relative *D*- and D_s-efficiency criteria

In this section, the designs were compared in terms of the (overall) D-efficiency defined in (7) for the full model (*D*), and D_s-efficiency defined in (8) for linear terms (D_l), quadratic terms (D_q), and bilinear terms (D_b), respectively at zero center point ($n_c = 0$). The design efficiencies in estimating a subset (D_s) of the model parameters are compared by dividing the model parameters into three groups: the linear parameters ($B_j, j = 1, ..., k$), the quadratic parameters ($B_{jj}, j = 1, ..., k$) and the bilinear parameters ($B_{ij}, 1 \leq i < j \leq k$). The $D_{-efficiency}$ for the full model and $D_{s-efficiency}$ for each subset (D_l, D_q and D_b) of the parameters of the model are computed, and the results are presented in Table 6 and Figures 1, 2, 3, and 4 respectively.

$$D_{-efficiency} = N^{-1}|\mathrm{X}^{\mathrm{T}}\mathrm{X}|^{1/\mathrm{p}} \quad (7)$$

and

$$D_{s-efficiency} = N^{-1}|\mathrm{X}_s^{\mathrm{T}}\mathrm{X}_s - \mathrm{X}_s^{\mathrm{T}}\mathrm{X}_{(\mathrm{s})}(\mathrm{X}_{(s)}^{\mathrm{T}}\mathrm{X}_{(\mathrm{s})})^{-1}\mathrm{X}_{(s)}^{\mathrm{T}}\mathrm{X}_s|^{1/p_s} \quad (8)$$

where s is any of linear (l), quadratic (q), and bilinear (b); p_s is the number of parameters in s; p is the number of parameters for the full model and for an N-point design, **X** is the model matrix of the model defined in (2).

Table 6: D_s-efficiencies of different designs with n_c = 0

	Design	n	D	D_l	D_q	D_b
	CCD	26	1.7764	2.4323	3.3214	2.4132
$k = 5$	**SCD**	22	1.6321	1.8698	2.9425	2.7142
	OACD	34	1.5623	2.0082	2.8848	1.4519
	SOACD	34	1.5643	2.0081	2.8845	1.4521
	CCD	44	1.6031	2.0453	3.9412	2.2459
$k = 6$	**SCD**	28	1.5571	1.7651	3.2014	2.5343
	OACD	50	1.4845	2.1231	3.4752	1.2341
	SOACD	50	1.4961	2.1256	3.4781	1.2024
	CCD	78	1.6267	1.9402	5.0124	2.1285
$k = 7$	**SCD**	38	1.5352	1.5690	3.7219	2.6894
	OACD	50	1.7767	2.3024	4.2003	1.4862
	SOACD	50	1.7920	2.3251	4.2429	1.5342
	CCD	80	1.5408	2.1754	5.1142	2.0567
$k = 8$	**SCD**	52	1.5056	1.8422	4.5012	2.2421
	OACD	59	1.7589	2.2799	3.8663	1.5543
	SOACD	59	1.7832	2.3215	3.8921	1.5821
	CCD	146	1.6402	2.1934	5.5442	1.8872
$k = 9$	**SCD**	58	1.4823	1.8380	4.3391	2.1241
	OACD	59	1.8673	2.3468	4.4907	1.6763
	SOACD	59	1.9309	2.4021	4.5281	1.7201

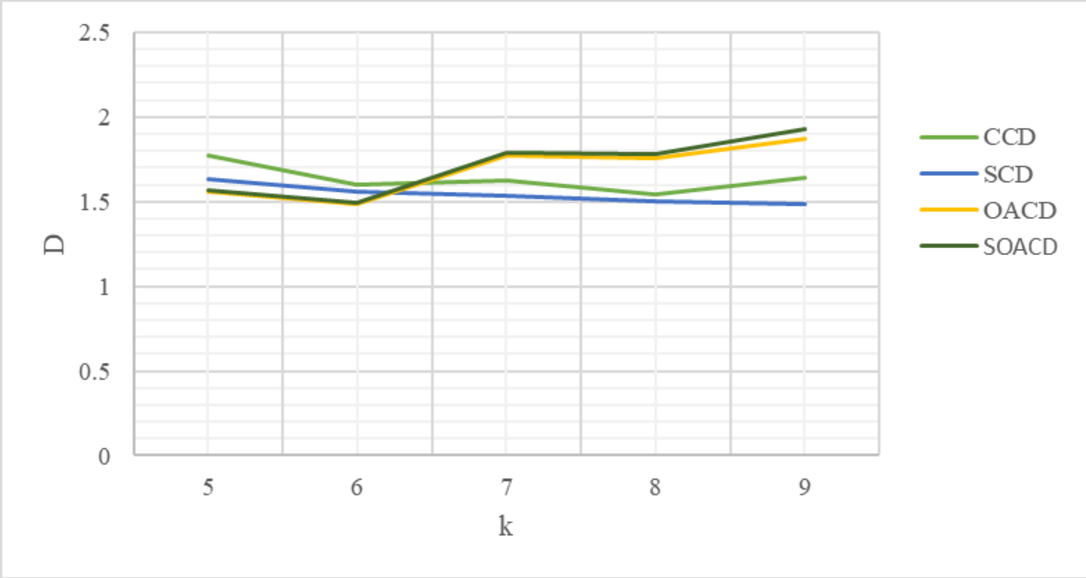

Figure 1 :Comparison of D-efficiecny for different designs

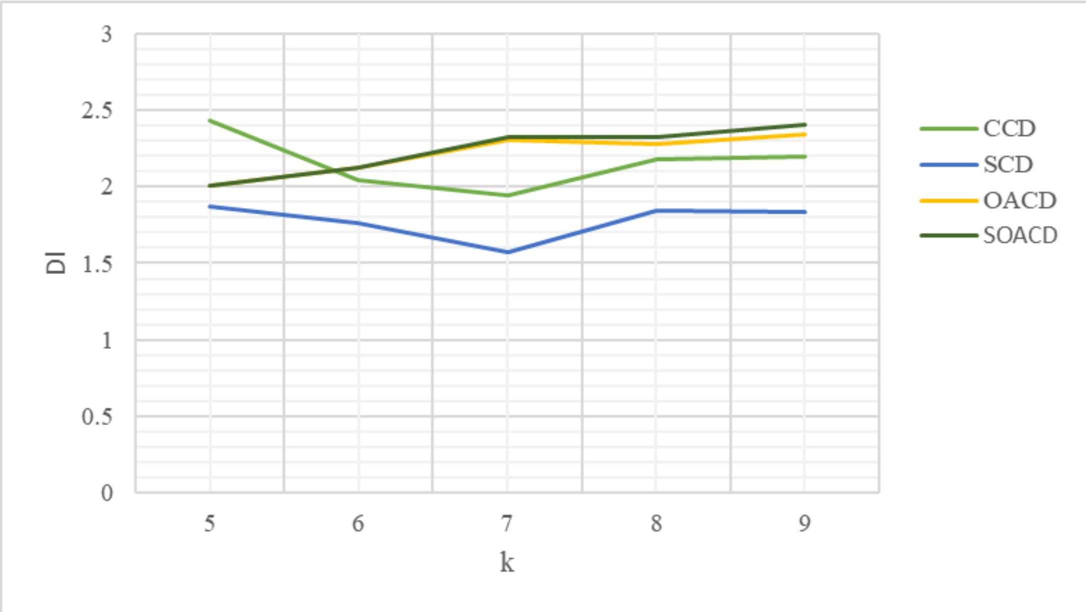

Figure 2 :Comparison of D_I-efficiency for different designs

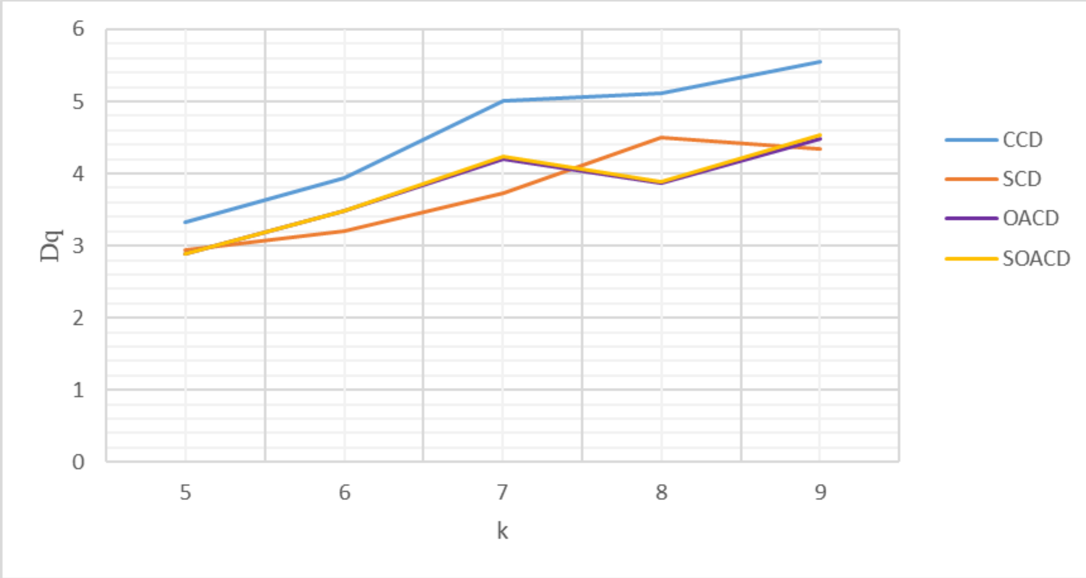

Figure 3 : Comparison of D_q-efficiency for different designs

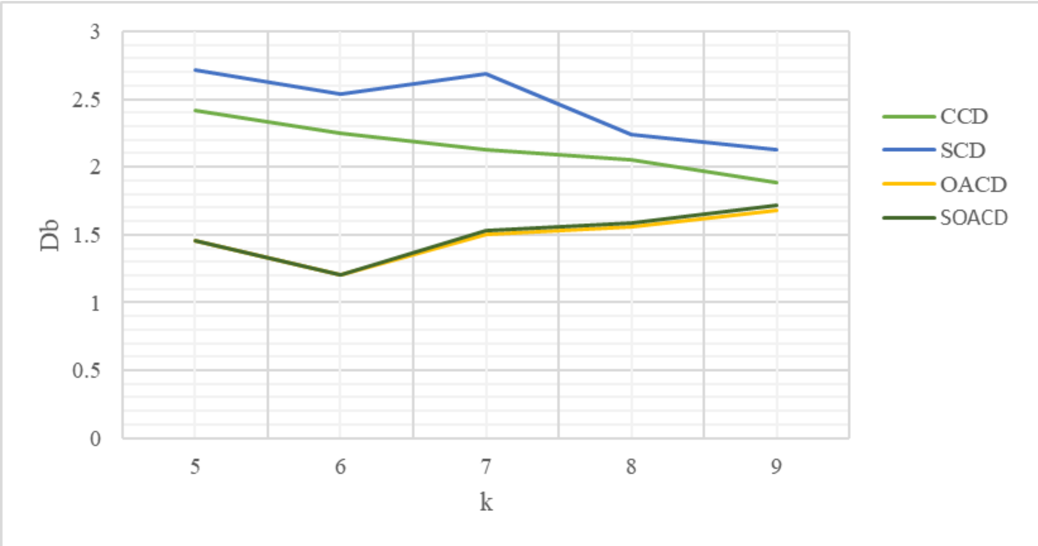

Figure 4 : Comparison of D_b-efficiency for different designs

In Figure 1, it can be seen that CCD has highest *D*-efficiency followed by SCD, OACD and SOACD at $k < 6$. As the number of factors increases $k > 6$, the efficiency values of the full model for CCDs and SCDs decreases while OACDs and SOACDs increases slightly. However, from Table 6 as the factors increases it can be observed that the run size of OACDs and SOACDs is lesser than CCDs but larger than of SCDs. Furthermore, the *D*-efficiency for SOACDs at k = 7, 8, and 9 are higher than that of CCDs, SCDs and OACDs.

Figure 2 compares the D_l-efficiencies for estimating the linear parameters. It can be seen that at $k < 6$ the D_l-efficiency of CCD performs better but as $k > 6$, SOACDs have the best D_l- efficiencies.
Figure 3 shows the D_q-efficiencies for comparing the quadratic terms. It can be observed that CCDs have the highest D_q-efficiencies followed by SOACDs except at $k = 5$ where SCDs and OACDs are greater. This is follows the note given by Chen et al. (2017), that more design points located at the corners will lead to higher D-, D_l- and D_b- efficiencies while more design points located at the mid-sides and center will increase the quadratic efficiency. Table 6 reveals that as the number of factors increases from 6, the D_q-efficiencies for SOACDs are higher than that of OACDs. The performance of the D_b-efficiencies for estimating the bilinear terms is shown in Figure 4. It can be observed that as the number of factors increases the performance of the CCDs and SCDs decreases while that of the OACDs and SOACDs increases respectively. Moreover, in Table 6, the SOACDs have the best D_b-efficiencies in some cases for $5 \leq k \leq 9$ as the factors increases.

5 Conclusion

In this paper, space-filling orthogonal-array based composite designs (SOACDs) were proposed using the maximin and the centered l_2-discrepancies. SOACDs were compared with OACDs, CCDs and SCDs under the overall relative D-efficiency and D_s-optimality. In terms of the relative D-efficiency, the SOACDs perform relatively better than CCDs, SCDs and OACDs in some cases for $5 \leq k \leq 11$ and $1 \leq n_c \leq 5$ as the factors increases. Under the D_s-optimality criterion, the SOACDs perform better in almost all the cases for estimating the full model, linear, quadratic and bilinear terms. On the run size basis, the SOACDs perform well as the number of factors increases due to significant reduction in run size compared to CCDs. Therefore, SOACD is recommended to be used in analyzing, estimating effects, and interactions when exploring high-dimensional computer simulations where there is considerable a prior uncertainty about the forms of response surfaces.

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